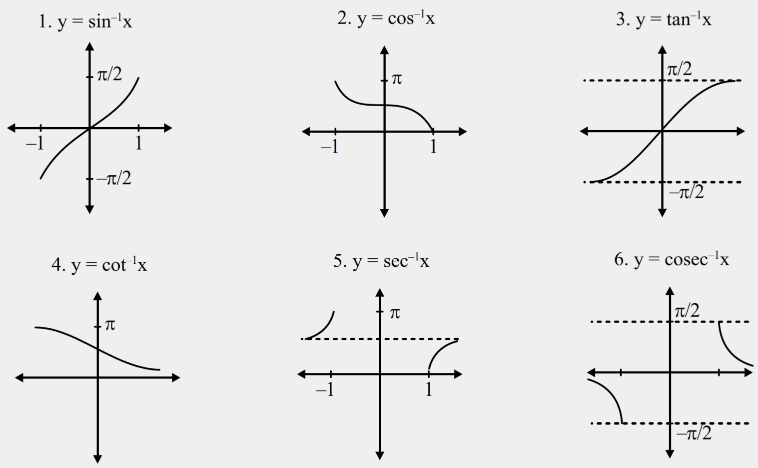
Inverse Trigonometric Functions

01

Inverse function	Domain	Principal Value Branch
$y=\sin^{-1}x$	$[-1, 1]$	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$
$y=\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$y=\text{cosec}^{-1}x$	$R-(-1,1)$	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$
$y=\sec^{-1}x$	$R-(-1,1)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$
$y=\tan^{-1}x$	R	$\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$
$y=\cot^{-1}x$	R	$(0,\pi)$

02

Graph



Properties Of Inverse Trigonometric Functions

03

Property -01

(i) $\sin^{-1}(\sin\theta)=\theta$ if $\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$

(ii) $\cos^{-1}(\cos\theta)=\theta$ if $0\le\theta\le\pi$

(iii) $\tan^{-1}(\tan\theta)=\theta$ if $\frac{-\pi}{2}<\theta<\frac{\pi}{2}$

(iv) $\cot^{-1}(\cot\theta)=\theta$ if $0<\theta<\pi$

(v) $\sec^{-1}(\sec\theta)=\theta$ if $0\le\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta\le\pi$

(vi) $\text{cosec}^{-1}(\text{cosec}\,\theta)=\theta$, if $-\frac{\pi}{2}\le\theta<0$ or $0<\theta\le\frac{\pi}{2}$

Property -02

(i) $\sin\left(\sin^{-1}x\right)=x$, if $-1\le x\le 1$

(ii) $\cos\left(\cos^{-1}x\right)=x$, if $-1\le x\le 1$

(iii) $\tan\left(\tan^{-1}x\right)=x$, if $-\infty<x<\infty$

(v) $\sec\left(\sec^{-1}x\right)=x$, if $-\infty<x\le-1$ or $1\le x<\infty$

(iv) $\cot\left(\cot^{-1}x\right)=x$, if $-\infty<x<\infty$

(vi) $\text{cosec}\left(\text{cosec}^{-1}x\right)=x$, if $-\infty<x\le-1$ or $1\le x<\infty$

Property -03

(i) $\sin^{-1}(-x)=-\sin^{-1}x$, if $-1\le x\le 1$

(ii) $\cos^{-1}(-x)=\pi-\cos^{-1}x$, if $-1\le x\le 1$

(iii) $\tan^{-1}(-x)=-\tan^{-1}x$, if $-\infty<x<\infty$

(iv) $\cot^{-1}(-x)=\pi-\cot^{-1}x$, if $-\infty<x<\infty$

(v) $\sec^{-1}(-x)=\pi-\sec^{-1}x$, if $-\infty<x\le-1$ or $1\le x<\infty$

(vi) $\text{cosec}^{-1}(-x)=-\text{cosec}^{-1}x$, if $-\infty<x\le-1$ or $1\le x<\infty$

Property -04

(i) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}, x\in[-1,1]$

(ii) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\quad x\in R$

(iii) $\sec^{-1}x+\text{cosec}^{-1}x=\frac{\pi}{2}, x\in(-\infty,-1]\cup[1,\infty)$

Property -05

(i) $\sin^{-1}x=\text{cosec}^{-1}\left(\frac{1}{x}\right), -1\le x\le 1-\{0\}$

(ii) $\text{cosec}^{-1}x=\sin^{-1}\left(\frac{1}{x}\right), x\in R-(-1,1)$

(iii) $\cos^{-1}x=\sec^{-1}\left(\frac{1}{x}\right), -1\le x\le 1-\{0\}$

(iv) $\sec^{-1}x=\cos^{-1}\left(\frac{1}{x}\right), x\in R-(-1,1)$

(v) $\tan^{-1}x=\cot^{-1}\left(\frac{1}{x}\right), x\in R-\{0\}$

(vi) $\tan^{-1}(1/x)=\begin{cases}\cot^{-1}x & \forall\ x>0\\ -\pi+\cot^{-1}x & \forall\ x<0\end{cases}$

Property -06

(i) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1$

$=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), x>0, y>0, xy>1$

$=-\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), x<0, y<0, xy>1$

(ii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right), xy>-1$

$=\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right), x>0, y>0, xy<-1$

$=-\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right)$, *if x < 0, y < 0 and xy > 1*

(iii) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$, if $x>0, y>0, z>0$ and $(xy+yz+zx)<1$, *if x < 0, y < 0 and xy > 1*

Property -07

(i) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$, if $x^2+y^2\le 1$ or if $xy<0$ and $x^2+y^2>1$; where $x,y\in[-1,1]$

$=\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$ if $0<x\le 1, 0<y\le 1$ and $x^2+y^2>1$

$=-\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$, if $-1\le x<0, 0<y\le 1$ and $x^2+y^2>1$

(ii) $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$, $xy>0$, $x^2+y^2>1$ or $x^2+y^2\le 1$

$=\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$, $0<x\le 1$, $-1\le y\le 0$, $x^2+y^2>1$

$=-\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$, $-1\le x<0$, $0<y\le 1$, $x^2+y^2>1$

Property -08

(i) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}$, $-1\le x, y\le 1, x+y\ge 0$

$=2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}$, $-1\le x, y\le 1, x+y<0$

(ii) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}$, $-1\le x, y\le 1, x\le y$

$=-\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}$, $-1\le y\le 0, 0<x\le 1, x>y$

Property -09

(i) $2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right), \frac{-1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$

$=\pi-\sin^{-1}\left(2x\sqrt{1-2^2}\right), \quad \frac{1}{\sqrt{2}}\le x\le 1.$

$=-\pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right), -1\le x\le\frac{-1}{\sqrt{2}}$

(ii) $3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right), -1/2\le x\le 1/2$

$=\pi-\sin^{-1}\left(3x-4x^3\right), \ 1/2<x\le 1$

$=-\pi-\mathrm{san}^{-1}\left(3x-4x^3\right), -1\le x<-1/2$

Property -10

(i) $2\cos^{-1}x=\cos^{-1}\left(2x^2-1\right), \quad 0\le x\le 1. =2\pi-\cos^{-1}\left(2x^2-1\right), -1\le x\le 0$

(ii) $3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right), \quad 1/2\le x\le 1$

$=2\pi-\cos^{-1}\left(4x^3-3x\right), -1/2\le x\le 1/2$

$=2\pi+\cos^{-1}\left(4x^3-3x\right), -1\le x\le -1/2$

Property -11

(i) $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right), \quad -1<x<1$

$=\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), \quad x>1$

$=-\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), \quad x<-1$

(ii) $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right), \quad -1\le x\le 1$

$=\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right), \ x>1$

$=-\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right), \ x<-1$

(iv) $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \ \frac{-1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

$=\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \ x>\frac{1}{\sqrt{3}}$

$=-\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \ x<\frac{-1}{\sqrt{3}}$

(iii) $2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \quad 0\le x<\infty$

$=-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), -\infty<x\le 0$

Property -12

(i) $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\mathrm{cosec}^{-1}\left(\frac{1}{x}\right), x>0$

$\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{\sqrt{1-x^2}}{x}=\cot^{-1}\frac{x}{\sqrt{1-x^2}}=\sec^{-1}\left(\frac{1}{x}\right)=\mathrm{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right), x>0$

(iii) $\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)=\cot^{-1}\left(\frac{1}{x}\right)=\sec^{-1}\left(\sqrt{1+x^2}\right)=\mathrm{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

Property -13

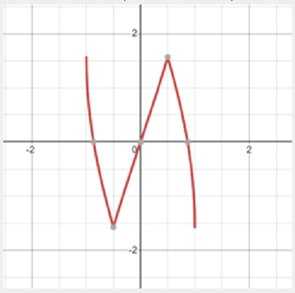
If $x_1, x_2, \ldots, x_n \in R$ then $\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n=$

$$\tan^{-1}\left(\frac{s_1-s_3+s_5-s_7+\cdots}{1-s_2+s_4-s_6+\cdots}\right)$$

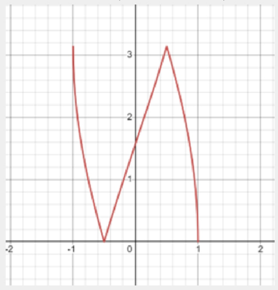
where, s_k = sum of products of $x_1, x_2, \ldots, k_n$ taken k at a time.

Some Important graphs

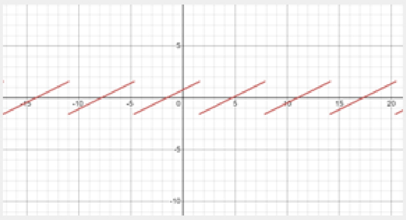
$\sin^{-1}\left(3x-4x^3\right)$



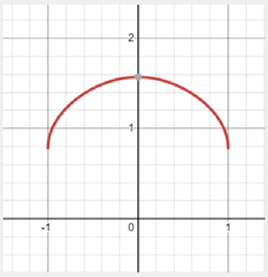
$\cos^{-1}\left(4x^3-3x\right)$



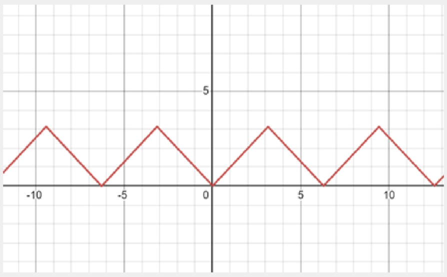
$\tan^{-1}\left(\sec x+\tan x\right)$



$\tan^{-1}\left(\frac{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)}\right) = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$



$\cos^{-1}\left(\cos x\right)$



$\sin^{-1}\left(\sin x\right)$